Range Theorems for Quantum Probability and Entanglement

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Abstract

We consider the set of all matrices of the form $p_{ij} = tr[W(E_i \otimes F_j)]$ where E_i, F_j are projections on a Hilbert space H, and W is some state on $H \otimes H$. We derive the basic properties of this set, compare it with the classical range of probability, and note how its properties may be related to a geometric measures of entanglement.

1 Introduction

Let n be a natural number and consider the space of $(n+1) \times (n+1)$ real matrices, which we shall denote by $\Re_{n+1}$.The indices of a matrix $(a_{ij}) \in \Re_{n+1}$ have range $0 \leq i, j \leq n$. My aim in this paper is to investigate the subset of $\Re_{n+1}$which is given by the following:

Definition 1 *$bell(n)$ is the set of all matrices $(p_{ij}) \in \Re_{n+1}$ with the following properties: $p_{00} = 1$ and there exist a finite dimensional Hilbert space H, projections $E_1, E_2, ..., E_n, F_1, F_2, ...F_n$ in H, and a statistical operator W on the tensor product $H \otimes H$ such that $p_{i0} = tr[W(E_i \otimes I)]$, $p_{0j} = tr[W(I \otimes F_j)]$, $p_{ij} = tr[W(E_i \otimes F_j)]$, for $i, j = 1, 2, ..., n$. Here I is the unit matrix on H. In case W is pure we shall say that (p_{ij}) has a pure state representation.*

Thus, $bell(n)$ is the range of probability values that states on a tensor product assign to quantum events. Of particular interest are the probability values assigned by entangled states which violate at least one Bell inequality. We shall compare these probability values with the values in $c(n)$, the classical range:

Definition 2 *$c(n)$ is the set of all matrices $(p_{ij}) \in \Re_{n+1}$ with the following properties: $p_{00} = 1$ and there exists a probability space (X, Σ, μ), events $A_1, A_2, ..., A_n$, $B_1, B_2, ...B_n \in \Sigma$, such that $p_{i0} = \mu(A_i)$, $p_{0j} = \mu(B_j)$, $p_{ij} = \mu(A_i \cap B_j)$ for $i, j = 1, 2, ..., n$.*

The set $c(n)$ has been completely characterized [1][2]. It is a polytope (the closed convex hull of finitely many matrices) whose vertices are the following: Let $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$, $(\delta_1, \delta_2, ...\delta_n) \in \{0,1\}^n$, be any two arbitrary n-vectors of zeroes and ones, define a matrix (u_{ij}) by $u_{00} = 1$, $u_{i0} = \varepsilon_i$, $u_{0j} = \delta_j$, $u_{ij} = \varepsilon_i \delta_j$ for $i, j = 1, 2, ..., n$. Each such choice defines a vertex of $c(n)$, altogether 2^{2n} vertices.

Every convex polytope in a linear space has a dual description, firstly in terms of its vertices and secondly in terms of its *facets*, linear inequalities which describe the half spaces that bound it. In the case of the correlation polytopes $c(n)$, the inequalities include the Bell inequalities, Clauser Horne inequalities, and other inequalities that arise in the study of entangled states. The investigation of these inequalities began a long time ago [3][4]. Deriving the description of a polytope in terms of inequalities from a description in terms of vertices is called *the hull problem.* It is algorithmically solvable, but in case of the correlation polytope the computational complexity is high [5]. For small n the problem can be solved fairly quickly on a personal computer [6][7]. In the case $n = 2$, the number of inequalities is 24 and they include the Clauser Horne inequalities, for $n = 3$ there are 684 inequalities!

In the next section I shall prove that $bell(n)$ is convex, and $bell(n) \supset c(n)$. A more detailed description is possible if we concentrate on a special subset of $bell(n)$: Denote by $bell_0(n)$ the subset of $\Re_{n+1}$ which is defined like $bell(n)$ but with the additional conditions on the marginals $p_{i0} = p_{0j} = 12$ for $i, j = 1, 2, ..., n$. Following [8] we shall give in the third section a simple characterization of the elements of $bell_0(n)$. Of course, $bell_0(n)$ is also convex and $bell_0(n) \supset c_0(n)$ where $c_0(n)$ is the subset of $c(n)$ defined by the same conditions.

It is also interesting to compare $bell(n)$ with another quantum range. Note that the tensor product plays the logical role of conjunction in quantum mechanics. Thus measuring $E_i \otimes F_j$ consists of measuring E_i "on the left" **and** F_j "on the right". This in analogy with the classical case where $A_i \cap B_j$ is the event A_i **and** B_j. However, the tensor product is not the most general form of conjunction in quantum mechanics. Thus, Birkhoff and von Neumann [9] suggested that the quantum analogue of **"and"** should be subspace intersection. This leads to the following definition

Definition 3 *$q(n)$ is the set of all matrices $(p_{ij}) \in \Re_{n+1}$ with the following properties: $p_{00} = 1$ and there exist a Hilbert space H, projections $E_1, E_2, ..., E_n$, $F_1, F_2, ...F_n$ in H which do not necessarily commute, and a statistical operator W on H such that $p_{i0} = tr(WE_i)$, $p_{0j} = tr(WF_j)$, $p_{ij} = tr[W(E_i \wedge F_j)]$, for $i, j = 1, 2, ..., n$. Here $E_i \wedge F_j$ is the projection on $E_i(H) \cap F_j(H)$.*

The set $q(n)$ has also been completely characterized [1][2]. It is convex but not (relatively) closed. Its closure in $\Re_{n+1}$ is a polytope with vertices (u_{ij}) which are all the zero-one matrices satisfying $u_{00} = 1$, $u_{i0} \geq u_{ij}$, $u_{0j} \geq u_{ij}$ for $i, j = 1, 2, ..., n$. We shall see that $q(n) \supset bell(n) \supset c(n)$.

Now, suppose that W is a *fixed* state on a tensor product $H \otimes H$. Let $n \geq 2$ be a natural number. *The trajectory of W in $\Re_{n+1}$*, denoted by $b(W, n)$, is the set of all matrices in $bell(n)$ which can be formed by applying W to arbitrary n projections on the "left" space and arbitrary n projections on the "right". Assume W is pure and $n \geq 2$. We shall see that if W is a product state then $b(W,n) \subset c(n)$, otherwise, $b(W,n) \nsubseteq c(n)$. Hence, if W is entangled then for all $n \geq 2$, parts of $b(W,n)$ lie outside of $c(n)$. The maximal distance between $b(W,n)$ and $c(n)$ may serve as a geometric measure of the entanglement of W.

Two remarks should be made at this point: (1) I have chosen the number of projections on the "left" to be identical to the number on the "right". There is no loss of generality in that, since the $n \times m$ case can be imbedded in the $\max(m,n) \times \max(m,n)$ case by adding zero projections. (2) many of the results that follow can be extended to multipartite cases.

2 The Set $bell(n)$

Theorem 4 *$bell(n)$ is convex and $q(n) \supset bell(n) \supset c(n)$.*

Proof. *Assume that $(p_{ij}),(q_{ij}) \in bell(n)$ and let $0 \leq \lambda \leq 1$, we shall show that $(\lambda p_{ij} + (1-\lambda)q_{ij}) \in bell(n)$. By assumption there exist a finite dimensional Hilbert space H, projections $E_0 = I, E_1, E_2, ..., E_n$, and $F_0 = I, F_1, F_2, ...F_n$ in H, and a statistical operator W on the tensor product $H \otimes H$ such that $p_{ij} = tr[W(E_i \otimes F_j)]$, for $i,j = 0,1,2,...,n$. Similarly, there exist a finite dimensional Hilbert space $H^{`}$, projections $E_0^{`} = I^{`}, E^{`}{}_1, E^{`}{}_2, ..., E^{`}{}_n$, and $F^{`}{}_0 = I^{`}, F^{`}{}_1, F^{`}{}_2, ...F^{`}{}_n$ in $H^{`}$, and a statistical operator $W^{`}$ on the tensor product $H^{`} \otimes H^{`}$ such that $q_{ij} = tr[W^{`}(E^{`}{}_i \otimes F^{`}{}_j)]$, for $i,j = 0,1,2,...,n$. (Here $I^{`}$ stands for the unit in $H^{`}$). Now, let $H^{``} = H \oplus H^{`}$ be the direct sum of $H$ and $H^{`}$, then $H^{``}$ is finite dimensional. define $E^{``}{}_i = E_i \oplus E^{`}{}_i$ and $F^{``}{}_j = F_j \oplus F^{`}{}_j$. Since W is a state in $H \otimes H$ it can be represented as $W = \sum_k \lambda_k |\Phi_k\rangle \langle \Phi_k|$ with $\lambda_k \geq 0, \sum_k \lambda_k = 1$, and $|\Phi_k\rangle$ unit vectors in H. Each vector $|\Phi_k\rangle$ can be identified as a vector $|\Phi_k^*\rangle$ on $H^{``} \otimes H^{``}$ as follows: If $|\Phi_k\rangle = \sum_l c_l |\alpha_l\rangle |\beta_l\rangle$ is the Schmidt decomposition of $|\Phi_k\rangle$, we shall identify it with $|\Phi_k^*\rangle = \sum_l c_l(|\alpha_k\rangle \oplus 0^{`}) \otimes (|\beta_l\rangle \oplus 0^{`}) \in H^{``} \otimes H^{``}$, where $0^{`}$ stands for the zero vector of $H^{`}$. Now identify W as the state $W^{\ *} = \sum_k \lambda_k |\Phi_k^*\rangle \langle \Phi_k^*|$ on $H^{``} \otimes H^{``}$.*

If W' is similarly represented on $H^{`} \otimes H^{`}$ in terms of vectors $|\Phi^{`}{}_k\rangle = \sum_k c_k' |\alpha_k'\rangle |\beta_k'\rangle$, we can identify each $|\Phi^{`}{}_k\rangle$ as a vector in $H^{``} \otimes H^{``}$, namely $\sum_k c^{`}{}_k(0 \oplus |\alpha^{`}{}_k\rangle) \otimes (0 \oplus |\beta^{`}{}_k\rangle)$. The state $W^{*\prime}$ is similarly identified as a state on $H^{``} \otimes H^{``}$. With this we can define the state $W^{``} = \lambda W^* + (1-\lambda)W^{*\prime}$ on $H^{``} \otimes H^{``}$. It is straightforward to see that $tr[W^{``}(E^{``}{}_i \otimes F^{``}{}_j)] = \lambda p_{ij} + (1-\lambda)q_{ij}$.

To prove that $bell(n) \supset c(n)$ let $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n), (\delta_1, \delta_2, ...\delta_n) \in \{0,1\}^n$ be two zero-one vectors. Let H be an arbitrary Hilbert space. Define projections on H by

$$F_j = \begin{cases} 0 & \delta_j = 0 \\ I & \delta_j = 1 \end{cases} \quad (1)$$

$$E_0 = F_0 = I \qquad E_i = \begin{cases} 0 & \varepsilon_i = 0 \\ I & \varepsilon_i = 1 \end{cases} \qquad F_j = \begin{cases} 0 & \delta_j = 0 \\ I & \delta_j = 1 \end{cases}$$

and let W be any state. Then $tr[W(E_i \otimes I)] = \varepsilon_i$, $tr[W(I \otimes F_j)] = \delta_j$, $tr[W(E_i \otimes F_j)] = \varepsilon_i \delta_j$. Hence, $bell(n)$ contains the vertices of $c(n)$. Since $bell(n)$ is convex it contains all convex combinations of the vertices. Therefore, $bell(n) \supset c(n)$. The other inclusion $q(n) \supset bell(n)$ is trivial, since $E_i \otimes F_j = (E_i \otimes I) \wedge (I \otimes F_j)$.
■

We shall denote by $bell_+(n)$ the set of all matrices $(q_{ij}) \in \Re_{n+1}$ with the following property: there exist a finite dimensional Hilbert space H, semi-definite operators $A_0 = I, A_1, A_2, ..., A_n$, $B_0 = I, B_1, B_2, ...B_n$ in H, with $spectrum[A_i] \subset [0,1]$, $spectrum[B_j] \subset [0,1]$, and a statistical operator W on the tensor product $H \otimes H$ such that $q_{ij} = tr[W(A_i \otimes B_j)]$, for $i, j = 0, 1, ..., n$. Obviously, $bell(n) \subseteq bell_+(n)$. It is easy to see, using identical technique to that of theorem 4, that $bell_+(n)$ is convex.

Theorem 5 $bell(n) = bell_+(n)$

Proof. Assume $q_{ij} = tr[W(A_i \otimes B_j)]$. If W is a mixture, $W = \sum_k \lambda_i |\Phi_k\rangle \langle\Phi_k|$, $\lambda_k \geq 0$, $\sum_k \lambda_k = 1$ then $q_{ij} = tr[W(A_i \otimes B_j)] = \sum_k \lambda_k \langle\Phi_k| A_i \otimes B_j |\Phi_k\rangle$ is a convex combination of elements of $bell_+(n)$ that have a pure state representation. Hence we can assume that W is pure.

If $q_{ij} = \langle\Phi| A_i \otimes B_j |\Phi\rangle$ and (at least) one of the A_i's or B_j's is not a projection operator then (q_{ij}) is a convex combination. Suppose, for example, that A_1 is not a projection operator. Then by the spectral theorem we can write $A_1 = \sum_{k=1}^{l} \eta_k E^k$ with $1 > \eta_1 > \eta_2 > ... > \eta_l > 0$ and E^k are pairwise orthogonal projections, $E^k E^r = E^r E^k = 0$. Hence for $j = 0, 1, ..., n$

$$\begin{aligned} q_{1j} = \langle\Phi| A_1 \otimes B_j |\Phi\rangle = \eta_l \langle\Phi| (E^1 + E^2 + ... + E^l) \otimes B_j |\Phi\rangle + \\ (\eta_{l-1} - \eta_l) \langle\Phi| (E^1 + E^2 + ... + E^{l-1}) \otimes B_j |\Phi\rangle + ... \\ ... + (\eta_1 - \eta_2) \langle\Phi| E^1 \otimes B_j |\Phi\rangle \end{aligned}$$

Note that $\eta_l + (\eta_{l-1} - \eta_l) + ... + (\eta_1 - \eta_2) = \eta_1 \leq 1$, also $E^1 + E^2 + ... + E^k$ are projection operators. Now, for $k = 1, 2, ..., l$ define

$$, \quad B_j^k = B_j (2)$$

$$A_0^k = B_0^k = I, \quad A_i^k = \begin{cases} E^1 + E^2 + ... + E^k & i = 1 \\ A_i & i > 1 \end{cases}, \quad B_j^k = B_j$$

Also put

$$, \quad B_j^{l+1} = B_j (3)$$

$$A_0^{l+1} = B_0^{l+1} = I, \quad A_i^{l+1} = \begin{cases} 0 & i = 1 \\ A_i & i > 1 \end{cases}, \quad B_j^{l+1} = B_j$$

then

$$\begin{aligned} q_{ij} &= (\eta_1-\eta_2)\langle\Phi|\,A_1^1\otimes B_j^1\,|\Phi\rangle + ... + (\eta_{l-1}-\eta_l)\langle\Phi|\,A_1^{l-1}\otimes B_j^{l-1}\,|\Phi\rangle + \\ &\quad +\eta_l\langle\Phi|\,A_1^l\otimes B_j^l\,|\Phi\rangle + (1-\eta_1)\langle\Phi|\,A_1^{l+1}\otimes B_j^{l+1}\,|\Phi\rangle \end{aligned}$$

Combining the two stages we see that every element of $bell_+(n)$ -and thus also of $bell(n)$- can be written as a convex combination of matrices of the form $e_{ij} = \langle\Phi|\,E_i\otimes F_j\,|\Phi\rangle$. Such matrices belong to $bell(n)$, hence, by convexity $bell(n) = bell_+(n)$. ■

Let H be a Hilbert space of a finite dimension m. Let a unit vector $|\Phi\rangle$ in $H\otimes H$ be given in the Schmidt form $|\Phi\rangle = \sum_i c_i\,|\alpha_i\rangle\,|\beta_i\rangle$ where c_i are real and non-negative $\sum_j c_i^2 = 1$, and $\{|\alpha_i\rangle\}$, and $\{|\beta_i\rangle\}$, $i=1,2,...,m$ two orthonormal bases in H. If E,F are projections in H we have for $W = |\Phi\rangle\langle\Phi|$: $tr[W(E\otimes F)] = \langle\Phi|\,E\otimes F\,|\Phi\rangle = \sum\limits_{ij} c_ic_j\,\langle\alpha_i|\,E\,|\alpha_j\rangle\,\langle\beta_i|\,F\,|\beta_j\rangle$. Let C be the diagonal matrix with $c_1,c_2,...,c_m$ on the diagonal, put $E_{ij} = \langle\alpha_i|\,E\,|\alpha_j\rangle$, and $F_{ij} = \langle\beta_i|\,F\,|\beta_j\rangle$ then $tr(CECF) = \sum_{ij} c_ic_jE_{ij}F_{ji}$. Now, define $F^*_{ij} = F_{ji} = \langle\beta_j|\,F\,|\beta_i\rangle$ and note that F^* is also a projection since $(F^{*2})_{ik} = \sum_j F^*_{ij}F^*_{jk}$ $= \sum_j F_{ji}F_{kj} = \sum_j F_{kj}F_{ji} = (F^2)_{ki} = F_{ki} = F^*_{ik}$. Hence we can write $tr[W(E\otimes F)] = tr(CECF^*)$. Combining this fact with theorem 2 we have proved:

Corollary 6 *: If* $(p_{ij})\in bell(n)$ *it can be represented as a convex combination of matrices of the form* $e_{ij} = tr(CE_iCF_j)$ *where* C *is diagonal positive and* $tr(C^2) = 1$

For the sake of completeness we should say something about the closure (in the Euclidean topology) of $bell(n)$, call it $\overline{bell}(n)$. If we could find a natural number N, such that every $(p_{ij})\in \overline{bell}(n)$ can be represented on a Hilbert space of dimension $\le N$, then $bell(n) = \overline{bell}(n)$. Moreover, then the extreme points of $bell(n)$ must have the form $e_{ij} = \langle\Phi_s|\,E_i\otimes F_j\,|\Phi_s\rangle$. However, I was not able to prove that. (We shall see below that in $bell_0(n)$ there is such a bound on the dimension). What we can show, however, is that the elements of $\overline{bell}(n)$ have a representation on a (possibly infinite dimesional) Hilbert space:

Theorem 7 *If* $(p_{ij})\in\overline{bell}(n)$ *then there exist a Hilbert space* H*, projections* $E_0 = I, E_1, E_2, ..., E_n$*, and* $F_0 = I, F_1, F_2, ...F_n$ *in* H*, and a statistical operator* W *on the tensor product* $H\otimes H$ *such that* $p_{ij} = tr[W(E_i\otimes F_j)]$*, for* $i,j = 0,1,2,...,n$.

Proof. *Let* $\{(p^k_{ij})\}_{k=1,2,...}$ *be a sequence of elements of bell(n) which converges in the Euclidean topology to* (p_{ij})*. This means, in particular, that* $p^k_{ij}\to p_{ij}$ *for all* i,j *and therefore also that* $\lim\limits_{K\to\infty} K^{-1}\sum_{n=1}^k p^k_{ij} = p_{ij}$*. By assumption, for each* k*, there is a finite dimensional Hilbert space* H_k *projections* $E^k_0 = I, E^k_1, E^k_2, ..., E^k_n$*, and* $F^k_0 = I, F^k_1, F^k_2, ...F^k_n$ *in* H_k *and a statistical operator* W_k *on the tensor product* $H_k\otimes H_k$ *such that* $p^k_{ij} = tr[W(E^k_i\otimes F^k_j)]$*, for* $i,j = 0,1,2,...,n$*. Consider the space* $\mathbb{H}_K = H_1\otimes H_2\otimes ...\otimes H_K$ *, the projections* $\mathbb{E}^K_i = E^1_i\otimes E^2_i\otimes ...\otimes E^K_i$*,* $\mathbb{F}^K_j = F^1_j\otimes F^2_j\otimes ...\otimes F^K_j$ *and the* *state* $\mathbb{W}_K$ *on* $\mathbb{H}_K \otimes \mathbb{H}_K$ *defined as follows: If* $W_r = \sum_{kl} \lambda_{kl} |\alpha_k\rangle \langle\alpha_k| \otimes |\beta_l\rangle \langle\beta_l|$ *put* $W_r^* = \sum_{kl} \lambda_{kl}(I \otimes ... \otimes \underset{r}{|\alpha_k\rangle \langle\alpha_k|} \otimes ... \otimes I) \otimes (I \otimes ... \otimes \underset{r}{|\beta_l\rangle \langle\beta_l|} \otimes ... \otimes I)$ *and* $\mathbb{W}_K = K^{-1}(W_1^* + W_2^* + ... + W_K^*)$. *Then it is easy to see that* $tr[\mathbb{W}_K(\mathbb{E}_i^K \otimes \mathbb{F}_j^K)] = K^{-1}\sum_{n=1}^{k} p_{ij}^k$. *Now , by a standard procedure [10][11] we can take the infinite tensor product limit* $\mathbb{H}_\infty$ *the limits* $\mathbb{E}_i^\infty$, *and* $\mathbb{F}_j^\infty$ *and the limit* $\mathbb{W}_\infty$ *on* $\mathbb{H}_\infty \otimes \mathbb{H}_\infty$ *with* $tr[\mathbb{W}_\infty(\mathbb{E}_i^\infty \otimes \mathbb{F}_j^\infty)] = p_{ij}$. ■

3 The set $bell_0(n)$

Cirel'son (also spelled Tsirelson) [8] considered the range of the expectation values $s_{ij} = tr[W(A_i \otimes B_j)]$ of operators A_i, B_j which satisfy $spectrum[A_i] \subset [-1,1]$, $spectrum[B_j] \subset [-1,1]$. The (s_{ij}) is taken as an $n \times n$ matrix and we do not include the marginal values $tr[W(I \otimes B_j)]$, and $tr[W(A_i \otimes I)]$. This is a crucial point, as we shall see later. Cirel'son's theorem is:

Theorem 8 *The following conditions on an* $n \times n$ *matrix* (s_{ij}) *are equivalent:*

a. There exists a Hilbert space H*, Hermitian operators* $A_1, A_2, ...A_n$*,* $B_1, B_2, ...B_n$*, and a state* W *on* $H \otimes H$ *such that* $spectrum[A_i] \subset [-1,1]$*,* $spectrum[B_j] \subset [-1,1]$ *and* $s_{ij} = tr[W(A_i \otimes B_j)]$ *for* $i,j = 1,2,...,n$.

b. The same as in 1, but with the additional conditions: $A_i^2 = I$*,* $B_j^2 = I$*,* $tr[W(A_i \otimes I)] = 0$*,* $tr[W(I \otimes B_j)] = 0$*,* $A_{i_1}A_{i_2} + A_{i_2}A_{i_1}$ *is proportional to* I *for all* $i_1, i_2 = 1,2,...,n$*,* $B_{j_1}B_{j_2} + B_{j_2}B_{j_1}$ *is proportional to* I *for all* $j_1, j_2 = 1,2,...,n$*, and* $\dim H \leq 2^{[n+12]}$.

c. There exist unit vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ *and* $\mathbf{y}_1, \mathbf{y}_2, ...\mathbf{y}_n$ *in the* $2n$*-dimensional real space* $\mathbb{R}^{2n}$ *such that* $s_{ij} = \mathbf{x}_i \cdot \mathbf{y}_j$.

Call the set defined by the conditions of theorem 4 $tsirelson(n)$.To see its connection with $bell_0(n)$ consider the second characterization in theorem 4. If A_i satisfies $A_i^2 = I$ then by the spectral theorem we can write $A_i = E_i - E_i^\perp$ where E_i is a projection operator, and $E_i^\perp$ is the projection on the subspace orthogonal to $E_i(H)$. Similarly we can write $B_j = F_j - F_j^\perp$. Now, from $tr[W(A_i \otimes I)] = 0$ and the fact that $E_i + E_i^\perp = I$ we conclude that $tr[W(E_i \otimes I)] = tr[W(E_i^\perp \otimes I)] = 12$. Similarly, $tr[W(I \otimes F_j)] = tr[W(I \otimes F_j^\perp)] = 12$. Denote $p_{ij} = tr[W(E_i \otimes F_j)]$ then

$$s_{ij} = tr[W(A_i \otimes B_j)] = tr[W(E_i - E_i^\perp) \otimes (F_j - F_j^\perp)] = 4p_{ij} - 1$$

Since $tr[W(E_i^\perp \otimes F_j)] = 12 - p_{ij}$, $tr[W(E_i \otimes F_j^\perp)] = 12 - p_{ij}$, $tr[W(E_i^\perp \otimes F_j^\perp)] = p_{ij}$. Therefore, the map $s_{ij} \longmapsto 14(s_{ij}+1)$ maps $tsirelson(n)$ to $bell_0(n)$.

Conversely let $p_{ij} = tr[W(E_i \otimes F_j)]$ be any element of $bell(n)$ (note! not necessarily $bell_0(n)$). Put $A_i = E_i - E_i^\perp$ and $B_j = F_j - F_j^\perp$, then by theorem 4a $s_{ij} = tr[W(A_i \otimes B_j)] \in tsirelson(n)$. Hence, the map $p_{ij} \longmapsto 4p_{ij} - 2p_{i0} -$

$2p_{0j}+1$ takes $bell(n)$ to $tsirelson(n)$ Combining the two maps $bell(n) \longmapsto tsirelson(n) \longmapsto bell_0(n)$ we see that $p_{i0} \longmapsto 12$, $p_{0j} \longmapsto 12$, $p_{ij} \longmapsto p_{ij} - 12p_{i0} - 12p_{0j} + 12$ maps $bell(n)$ to $bell_0(n)$. Altogether we have shown

Corollary 9 *(a) The set $bell_0(n)$ is convex and closed. If $(p_{ij}) \in bell_0(n)$ there is a Hilbert space H with $\dim H \leq 2^{[n+12]}$, projections $E_1, E_2, ..., E_n, F_1, F_2, ...F_n$ in H, and a state W such that $tr[W(E_i \otimes I)] = tr[W(I \otimes F_j)] = 12$ and $p_{ij} = tr[W(E_i \otimes F_j)]$ for $i,j = 1,2,...,n$. Moreover, we can assume that $E_{i_1}E_{i_2}^{\perp} + E_{i_2}E_{i_1}^{\perp}$ and $F_{j_1}F_{j_2}^{\perp} + F_{j_2}F_{j_1}^{\perp}$ are proportional to I, for all $i_1, i_2, j_1, j_2 = 1,2,..,n$.*

(b) If $(p_{ij}) \in bell_0(n)$ there exist unit vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ and $\mathbf{y}_1, \mathbf{y}_2, ...\mathbf{y}_n$ in the $2n$-dimensional real space $\mathbb{R}^{2n}$ such that $p_{ij} = 14(\mathbf{x}_i \cdot \mathbf{y}_j + 1)$.

4 Geometric Measures of Entanglement

In recent years there have been numerous attempts to quantify the "amount of entanglement" in a state defined on a tensor product of Hilbert spaces [12]. Most of these attempts are motivated by the concerns of quantum information theory. Here I shall take a different route. Roughly, the intuition is that the more entangled the state is the stronger the violation of (at least one) Bell inequality. For simplicity I shall concentrate on pure states.

Definition 10 *Let $W = |\Psi\rangle\langle\Psi|$ be a fixed state, $|\Psi\rangle = \sum_i c_i |\alpha_i\rangle |\beta_i\rangle$ its Schmidt decomposition. Then the trajectory of W on $\Re_{n+1}$, denoted by $b(W,n)$, is the set of all matrices $(p_{ij}) \in \Re_{n+1}$ that have the form $p_{ij} = \langle\Psi| E_i \otimes F_j |\Psi\rangle$, where $E_0 = I, E_1, E_2, ..., E_n$, and $F_0 = I, F_1, F_2, ...F_n$ are any projections in any finite dimensional Hilbert space H that contain $\{|\alpha_i\rangle\}$, and $\{|\beta_i\rangle\}$.*

The connection between the trajectory and the classical range $c(n)$ is given in the following.

Lemma 11 *If $W = |\Psi\rangle\langle\Psi|$ is a product state, $|\Psi\rangle = |\alpha\rangle|\beta\rangle$ then $b(W,n) \subset c(n)$ for all n. Conversely, if W is not a product state then $b(W,n) \nsubseteq c(n)$ for all $n \geq 2$.*

Proof. Suppose $|\Psi\rangle = |\alpha\rangle|\beta\rangle$ and let $E_0 = I, E_1, E_2, ..., E_n$, and $E_0 = I, E_1, E_2, ...F_n$ be any projections in H. Consider the unit square $[0,1]\times[0,1]$ in the real plane $\mathbb{R}^2$ as a probability space with Σ the algebra of Borel subsets and μ the uniform (Lebesgue) probability measure. Let A_i be the subset of $[0,1]\times[0,1]$ defined as $A_i = [0, \langle\alpha| E_i |\alpha\rangle] \times [0,1]$ similarly define $B_j = [0,1]\times[0, \langle\beta| F_j |\beta\rangle]$. Then $p_{i0} = \mu(A_i) = \langle\alpha| E_i |\alpha\rangle$, $p_{0j} = \mu(B_j) = \langle\beta| F_j |\beta\rangle$, $p_{ij} = \mu(A_i \cap B_j) = \langle\alpha| E_i |\alpha\rangle \langle\beta| F_j |\beta\rangle$ for $i,j = 1,2,...,n$. Hence $p_{ij} = \langle\Psi| E_i \otimes F_j |\Psi\rangle$ is an element of $c(n)$.

As for the converse, it follows from a theorem of Gisin and Peres[13]. They showed that if $|\Psi\rangle$ is not a product state then one can choose projections $E_0 = I, E_1, E_2$ and $E_0 = I, E_1, E_2$ such that $p_{ij} = \langle\Psi| E_i \otimes F_j |\Psi\rangle$ $ij = 0,1,2$ violate the Clauser-Horne inequality. This inequality is a facet inequality of $c(n)$ for all $n \geq 2$ [2]. Hence, $b(W,n) \nsubseteq c(n)$ for all $n \geq 2$. (It should be noted that Gisin and Peres use observables with eigenvalues ± 1. The transformation to projection operators is the same as in the previous section). ■

Let $\| \ \|$ be a norn defined on $\Re_{n+1}$, where $n \geq 2$ is fixed, and assume that $\| \ \|$ is continuous with respect to the Euclidean topology on $\Re_{n+1}$. Let $W = |\Psi\rangle \langle\Psi|$ be a pure state on $H \otimes H$, we shall define the entanglement measure associated with $\| \ \|$ to be

$$\mathcal{E}(W) = \sup_{(p_{ij}) \in b(W,n)} \min_{(q_{ij}) \in c(n)} \|(p_{ij}) - (q_{ij})\| \tag{1}$$

The minimum in (1) is obtained for each $(p_{ij}) \in b(W,n)$, because $\| \ \|$ is continuous and $c(n)$ compact in the Euclidean topology. From lemma it follows that $\mathcal{E}(W) = 0$ if, and only if W is a product state.

Problem 12 *Let* $|\Psi\rangle = \sum_{i=1}^{m} c_i |\alpha_i\rangle |\beta_i\rangle$ *and* $|\Phi\rangle = \sum_{i=1}^{m} d_i |\delta_i\rangle |\gamma_i\rangle$ *be the Schmidt decompositions of two unit vectors on* $H \otimes H$. *Assume* $c_1 \geq c_2 \geq ... \geq c_m \geq 0$, *and* $d_1 \geq d_2 \geq ... \geq d_m \geq 0$. *Recall that* $|\Psi\rangle$ *majorizes* $|\Phi\rangle$ *if* $\sum_{i=1}^{k} c_i^2 \geq \sum_{i=1}^{k} d_i^2$ *for all* $1 \leq k \leq m$, *in this case we shall denote* $|\Psi\rangle \succcurlyeq |\Phi\rangle$. *Under what conditions* $\mathcal{E}(W)$ *is monotone decreasing:* $|\Psi\rangle \succcurlyeq |\Phi\rangle$ *entails* $\mathcal{E}(|\Psi\rangle \langle\Psi|) \leq \mathcal{E}(|\Phi\rangle \langle\Phi|)$

Here the theorem of Nielsen may be helpful[12][14].

Problem 13 *Does any of the familiar entanglement measures, in particular von Neumann's entropy, have a geometric origin as above?*

I do not know the answer. A possible way to go is to use the uniqueness theorems [12][15], and try determine if there is a geometric measure which conforms with its conditions.

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